1. **a.** Let N stand for "News" and M stand for "Music." Then the listeners' behavior is given by the table

From:		
N	M	To:
.7	.6	N
.3	.4	M

so the stochastic matrix is $P = \begin{bmatrix} .7 & .6 \\ .3 & .4 \end{bmatrix}$.

b. Since 100% of the listeners are listening to news at 8: 15, the initial state vector is $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

c. There are two breaks between 8: 15 and 9: 25, so we calculate $\mathbf{x}_2$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .7 & .6 \\ .3 & .4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .7 & .6 \\ .3 & .4 \end{bmatrix} \begin{bmatrix} .7 \\ .3 \end{bmatrix} = \begin{bmatrix} .67 \\ .33 \end{bmatrix}$$

Thus 33% of the listeners are listening to news at 9: 25.

4. a. Let G stand for good weather, I for indifferent weather, and B for bad weather. Then the change in the weather is given by the table

From: G	I	B	To:
.6	.4	.4	G
.3	.3	.5	I
.1	.3	.1	B

so the stochastic matrix is $P = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix}$.

b. The initial state vector is $\begin{bmatrix} .5 \\ .5 \\ 0 \end{bmatrix}$. We calculate $\mathbf{x}_1$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix} \begin{bmatrix} .5 \\ .5 \\ 0 \end{bmatrix} = \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix}$$

Thus the chance of bad weather tomorrow is 20%.

c. The initial state vector is $\mathbf{x}_0 = \begin{bmatrix} 0 \\ .4 \\ .6 \end{bmatrix}$. We calculate $\mathbf{x}_2$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix} \begin{bmatrix} 0 \\ .4 \\ .6 \end{bmatrix} = \begin{bmatrix} .4 \\ .42 \\ .18 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix} \begin{bmatrix} .4 \\ .42 \\ .18 \end{bmatrix} = \begin{bmatrix} .48 \\ .336 \\ .184 \end{bmatrix}$$

Thus the chance of good weather on Wednesday is 48%.

14. From Exercise 4, $P = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix}$, so $P - I = \begin{bmatrix} -.4 & .4 & .4 \\ .3 & -.7 & .5 \\ .1 & .3 & -.9 \end{bmatrix}$. Solving $(P - I)\mathbf{x} = \mathbf{0}$ by row reducing the augmented matrix gives

$$\begin{bmatrix} -.4 & .4 & .4 & 0 \\ .3 & -.7 & .5 & 0 \\ .1 & .3 & -.9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ sum to 6, multiply by 1/6 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/6 \end{bmatrix} \approx \begin{bmatrix} .5 \\ .333 \\ .167 \end{bmatrix}$. Thus in the long run the chance that a day has good weather is 50%.

16. [M] Let $P = \begin{bmatrix} .90 & .01 & .09 \\ .01 & .90 & .01 \\ .09 & .09 & .90 \end{bmatrix}$, so $P - I = \begin{bmatrix} -.10 & .01 & .09 \\ .01 & -.10 & .01 \\ .09 & .09 & -.1 \end{bmatrix}$. Solving $(P - I)\mathbf{x} = \mathbf{0}$ by row reducing the augmented matrix gives

$$\begin{bmatrix} -.10 & .01 & .09 & 0 \\ .01 & -.10 & .01 & 0 \\ .09 & .09 & -.1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -.919192 & 0 \\ 0 & 1 & -.191919 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} .919192 \\ .191919 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} .919192 \\ .191919 \\ 1 \end{bmatrix}$. Since the entries in $\begin{bmatrix} .919192 \\ .191919 \\ 1 \end{bmatrix}$ sum to 2.111111, multiply by 1/2.111111 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix} .435407 \\ .090909 \\ .473684 \end{bmatrix}$. Thus on a typical day, about (.090909)(2000) = 182 cars will be rented or available from the downtown location.

18. If $\alpha = \beta = 0$ then $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Notice that $P\mathbf{x} = \mathbf{x}$ for any vector $\mathbf{x}$ in ^{2}, and that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are two linearly independent steady-state vectors in this case.

If $\alpha \neq 0$ or $\beta \neq 0$, we solve $(P - I)\mathbf{x} = \mathbf{0}$ where $P - I = \begin{bmatrix} -\alpha & \beta \\ \alpha & -\beta \end{bmatrix}$. Row reducing the augmented matrix gives

$$\begin{bmatrix} -\alpha & \beta & 0 \\ \alpha & -\beta & 0 \end{bmatrix} \sim \begin{bmatrix} \alpha & -\beta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\alpha x_1 = \beta x_2$, and one possible solution is to let $x_1 = \beta,\ x_2 = \alpha$. Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$. Since the entries in $\begin{bmatrix} \beta \\ \alpha \end{bmatrix}$ sum to $\alpha + \beta$, multiply by $1/(\alpha + \beta)$ to obtain the steady-state vector $\mathbf{q} = \frac{1}{\alpha + \beta}\begin{bmatrix} \beta \\ \alpha \end{bmatrix}$.